

FUNCTIONS DEFINED BY IMPROPER INTEGRALS

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This is a supplement to the author's

[Introduction to Real Analysis](#)

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1 Foreword

This is a revised version of Section 7.5 of my *Advanced Calculus* (Harper & Row, 1978). It is a supplement to my textbook *Introduction to Real Analysis*, which is referenced several times here. You should review Section 3.4 (Improper Integrals) of that book before reading this document.

2 Introduction

In Section 7.2 (pp. 462–484) we considered functions of the form

$$F(y) = \int_a^b f(x, y) dx, \quad c \leq y \leq d.$$

We saw that if f is continuous on $[a, b] \times [c, d]$, then F is continuous on $[c, d]$ (Exercise 7.2.3, p. 481) and that we can reverse the order of integration in

$$\int_c^d F(y) dy = \int_c^d \left(\int_a^b f(x, y) dx \right) dy$$

to evaluate it as

$$\int_c^d F(y) dy = \int_a^b \left(\int_c^d f(x, y) dy \right) dx$$

(Corollary 7.2.3, p. 466).

Here is another important property of F .

Theorem 1 *If f and f_y are continuous on $[a, b] \times [c, d]$, then*

$$F(y) = \int_a^b f(x, y) dx, \quad c \leq y \leq d, \tag{1}$$

is continuously differentiable on $[c, d]$ and $F'(y)$ can be obtained by differentiating (1) under the integral sign with respect to y ; that is,

$$F'(y) = \int_a^b f_y(x, y) dx, \quad c \leq y \leq d. \tag{2}$$

Here $F'(a)$ and $f_y(x, a)$ are derivatives from the right and $F'(b)$ and $f_y(x, b)$ are derivatives from the left.

Proof If y and $y + \Delta y$ are in $[c, d]$ and $\Delta y \neq 0$, then

$$\frac{F(y + \Delta y) - F(y)}{\Delta y} = \int_a^b \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} dx. \tag{3}$$

From the mean value theorem (Theorem 2.3.11, p. 83), if $x \in [a, b]$ and $y, y + \Delta y \in [c, d]$, there is a $y(x)$ between y and $y + \Delta y$ such that

$$f(x, y + \Delta y) - f(x, y) = f_y(x, y) \Delta y = f_y(x, y(x)) \Delta y + (f_y(x, y(x)) - f_y(x, y)) \Delta y.$$

From this and (3),

$$\left| \frac{F(y + \Delta y) - F(y)}{\Delta y} - \int_a^b f_y(x, y) dx \right| \leq \int_a^b |f_y(x, y(x)) - f_y(x, y)| dx. \quad (4)$$

Now suppose $\epsilon > 0$. Since f_y is uniformly continuous on the compact set $[a, b] \times [c, d]$ (Corollary 5.2.14, p. 314) and $y(x)$ is between y and $y + \Delta y$, there is a $\delta > 0$ such that if $|\Delta y| < \delta$ then

$$|f_y(x, y) - f_y(x, y(x))| < \epsilon, \quad (x, y) \in [a, b] \times [c, d].$$

This and (4) imply that

$$\left| \frac{F(y + \Delta y) - F(y)}{\Delta y} - \int_a^b f_y(x, y) dx \right| < \epsilon(b - a)$$

if y and $y + \Delta y$ are in $[c, d]$ and $0 < |\Delta y| < \delta$. This implies (2). Since the integral in (2) is continuous on $[c, d]$ (Exercise 7.2.3, p. 481, with f replaced by f_y), F' is continuous on $[c, d]$. \square

Example 1 Since

$$f(x, y) = \cos xy \quad \text{and} \quad f_y(x, y) = -x \sin xy$$

are continuous for all (x, y) , Theorem 1 implies that if

$$F(y) = \int_0^\pi \cos xy dx, \quad -\infty < y < \infty, \quad (5)$$

then

$$F'(y) = - \int_0^\pi x \sin xy dx, \quad -\infty < y < \infty. \quad (6)$$

(In applying Theorem 1 for a specific value of y , we take $R = [0, \pi] \times [-\rho, \rho]$, where $\rho > |y|$.) This provides a convenient way to evaluate the integral in (6): integrating the right side of (5) with respect to x yields

$$F(y) = \frac{\sin xy}{y} \Big|_{x=0}^\pi = \frac{\sin \pi y}{y}, \quad y \neq 0.$$

Differentiating this and using (6) yields

$$\int_0^\pi x \sin xy dx = \frac{\sin \pi y}{y^2} - \frac{\pi \cos \pi y}{y}, \quad y \neq 0.$$

To verify this, use integration by parts. ■

We will study the continuity, differentiability, and integrability of

$$F(y) = \int_a^b f(x, y) dx, \quad y \in S,$$

where S is an interval or a union of intervals, and F is a convergent improper integral for each $y \in S$. If the domain of f is $[a, b) \times S$ where $-\infty < a < b \leq \infty$, we say that F is *pointwise convergent on S* or simply *convergent on S* , and write

$$\int_a^b f(x, y) dx = \lim_{r \rightarrow b-} \int_a^r f(x, y) dx \quad (7)$$

if, for each $y \in S$ and every $\epsilon > 0$, there is an $r = r_0(y)$ (which also depends on ϵ) such that

$$\left| F(y) - \int_a^r f(x, y) dx \right| = \left| \int_r^b f(x, y) dx \right| < \epsilon, \quad r_0(y) \leq y < b. \quad (8)$$

If the domain of f is $(a, b] \times S$ where $-\infty \leq a < b < \infty$, we replace (7) by

$$\int_a^b f(x, y) dx = \lim_{r \rightarrow a+} \int_r^b f(x, y) dx$$

and (8) by

$$\left| F(y) - \int_r^b f(x, y) dx \right| = \left| \int_a^r f(x, y) dx \right| < \epsilon, \quad a < r \leq r_0(y).$$

In general, pointwise convergence of F for all $y \in S$ does not imply that F is continuous or integrable on $[c, d]$, and the additional assumptions that f_y is continuous and $\int_a^b f_y(x, y) dx$ converges do not imply (2).

Example 2 The function

$$f(x, y) = ye^{-|y|x}$$

is continuous on $[0, \infty) \times (-\infty, \infty)$ and

$$F(y) = \int_0^\infty f(x, y) dx = \int_0^\infty ye^{-|y|x} dx$$

converges for all y , with

$$F(y) = \begin{cases} -1 & y < 0, \\ 0 & y = 0, \\ 1 & y > 0; \end{cases}$$

therefore, F is discontinuous at $y = 0$.

Example 3 The function

$$f(x, y) = y^3 e^{-y^2 x}$$

is continuous on $[0, \infty) \times (-\infty, \infty)$. Let

$$F(y) = \int_0^\infty f(x, y) dx = \int_0^\infty y^3 e^{-y^2 x} dx = y, \quad -\infty < y < \infty.$$

Then

$$F'(y) = 1, \quad -\infty < y < \infty.$$

However,

$$\int_0^\infty \frac{\partial}{\partial y} (y^3 e^{-y^2 x}) dx = \int_0^\infty (3y^2 - 2y^4 x) e^{-y^2 x} dx = \begin{cases} 1, & y \neq 0, \\ 0, & y = 0, \end{cases}$$

so

$$F'(y) \neq \int_0^\infty \frac{\partial f(x, y)}{\partial y} dx \quad \text{if } y = 0.$$

3 Preparation

We begin with two useful convergence criteria for improper integrals that do not involve a parameter. Consistent with the definition on p. 152, we say that f is locally integrable on an interval I if it is integrable on every finite closed subinterval of I .

Theorem 2 (Cauchy Criterion for Convergence of an Improper Integral I) Suppose g is locally integrable on $[a, b)$ and denote

$$G(r) = \int_a^r g(x) dx, \quad a \leq r < b.$$

Then the improper integral $\int_a^b g(x) dx$ converges if and only if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that

$$|G(r) - G(r_1)| < \epsilon, \quad r_0 \leq r, r_1 < b. \quad (9)$$

Proof For necessity, suppose $\int_a^b g(x) dx = L$. By definition, this means that for each $\epsilon > 0$ there is an $r_0 \in [a, b)$ such that

$$|G(r) - L| < \frac{\epsilon}{2} \quad \text{and} \quad |G(r_1) - L| < \frac{\epsilon}{2}, \quad r_0 \leq r, r_1 < b.$$

Therefore

$$\begin{aligned} |G(r) - G(r_1)| &= |(G(r) - L) - (G(r_1) - L)| \\ &\leq |G(r) - L| + |G(r_1) - L| < \epsilon, \quad r_0 \leq r, r_1 < b. \end{aligned}$$

For sufficiency, (9) implies that

$$|G(r)| = |G(r_1) + (G(r) - G(r_1))| \leq |G(r_1)| + |G(r) - G(r_1)| \leq |G(r_1)| + \epsilon,$$

$r_0 \leq r \leq r_1 < b$. Since G is also bounded on the compact set $[a, r_0]$ (Theorem 5.2.11, p. 313), G is bounded on $[a, b)$. Therefore the monotonic functions

$$\overline{G}(r) = \sup \{G(r_1) \mid r \leq r_1 < b\} \quad \text{and} \quad \underline{G}(r) = \inf \{G(r_1) \mid r \leq r_1 < b\}$$

are well defined on $[a, b)$, and

$$\lim_{r \rightarrow b-} \overline{G}(r) = \overline{L} \quad \text{and} \quad \lim_{r \rightarrow b-} \underline{G}(r) = \underline{L}$$

both exist and are finite (Theorem 2.1.11, p. 47). From (9),

$$\begin{aligned} |G(r) - G(r_1)| &= |(G(r) - G(r_0)) - (G(r_1) - G(r_0))| \\ &\leq |G(r) - G(r_0)| + |G(r_1) - G(r_0)| < 2\epsilon, \end{aligned}$$

so

$$\overline{G}(r) - \underline{G}(r) \leq 2\epsilon, \quad r_0 \leq r, r_1 < b.$$

Since ϵ is an arbitrary positive number, this implies that

$$\lim_{r \rightarrow b-} (\overline{G}(r) - \underline{G}(r)) = 0,$$

so $\overline{L} = \underline{L}$. Let $L = \overline{L} = \underline{L}$. Since

$$\underline{G}(r) \leq G(r) \leq \overline{G}(r),$$

it follows that $\lim_{r \rightarrow b-} G(r) = L$. □

We leave the proof of the following theorem to you (Exercise 2).

Theorem 3 (Cauchy Criterion for Convergence of an Improper Integral II) *Suppose g is locally integrable on $(a, b]$ and denote*

$$G(r) = \int_r^b g(x) dx, \quad a \leq r < b.$$

Then the improper integral $\int_a^b g(x) dx$ converges if and only if, for each $\epsilon > 0$, there is an $r_0 \in (a, b]$ such that

$$|G(r) - G(r_1)| < \epsilon, \quad a < r, r_1 \leq r_0.$$

To see why we associate Theorems 2 and 3 with Cauchy, compare them with Theorem 4.3.5 (p. 204)

4 Uniform convergence of improper integrals

Henceforth we deal with functions $f = f(x, y)$ with domains $I \times S$, where S is an interval or a union of intervals and I is of one of the following forms:

- $[a, b)$ with $-\infty < a < b \leq \infty$;
- $(a, b]$ with $-\infty \leq a < b < \infty$;
- (a, b) with $-\infty \leq a \leq b \leq \infty$.

In all cases it is to be understood that f is locally integrable with respect to x on I . When we say that the improper integral $\int_a^b f(x, y) dx$ has a stated property “on S ” we mean that it has the property for every $y \in S$.

Definition 1 *If the improper integral*

$$\int_a^b f(x, y) dx = \lim_{r \rightarrow b-} \int_a^r f(x, y) dx \quad (10)$$

converges on S , it is said to converge uniformly (or be uniformly convergent) on S if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that

$$\left| \int_a^b f(x, y) dx - \int_a^r f(x, y) dx \right| < \epsilon, \quad y \in S, \quad r_0 \leq r < b,$$

or, equivalently,

$$\left| \int_r^b f(x, y) dx \right| < \epsilon, \quad y \in S, \quad r_0 \leq r < b. \quad (11)$$

The crucial difference between pointwise and uniform convergence is that $r_0(y)$ in (8) may depend upon the particular value of y , while the r_0 in (11) does not: one choice must work for all $y \in S$. Thus, uniform convergence implies pointwise convergence, but pointwise convergence does not imply uniform convergence.

Theorem 4 (Cauchy Criterion for Uniform Convergence I) *The improper integral in (10) converges uniformly on S if and only if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that*

$$\left| \int_r^{r_1} f(x, y) dx \right| < \epsilon, \quad y \in S, \quad r_0 \leq r, r_1 < b. \quad (12)$$

Proof Suppose $\int_a^b f(x, y) dx$ converges uniformly on S and $\epsilon > 0$. From Definition 1, there is an $r_0 \in [a, b)$ such that

$$\left| \int_r^b f(x, y) dx \right| < \frac{\epsilon}{2} \quad \text{and} \quad \left| \int_{r_1}^b f(x, y) dx \right| < \frac{\epsilon}{2}, \quad y \in S, \quad r_0 \leq r, r_1 < b. \quad (13)$$

Since

$$\int_r^{r_1} f(x, y) dx = \int_r^b f(x, y) dx - \int_{r_1}^b f(x, y) dx,$$

(13) and the triangle inequality imply (12).

For the converse, denote

$$F(y) = \int_a^r f(x, y) dx.$$

Since (12) implies that

$$|F(r, y) - F(r_1, y)| < \epsilon, \quad y \in S, \quad r_0 \leq r, r_1 < b, \quad (14)$$

Theorem 2 with $G(r) = F(r, y)$ (y fixed but arbitrary in S) implies that $\int_a^b f(x, y) dx$ converges pointwise for $y \in S$. Therefore, if $\epsilon > 0$ then, for each $y \in S$, there is an $r_0(y) \in [a, b)$ such that

$$\left| \int_r^b f(x, y) dx \right| < \epsilon, \quad y \in S, \quad r_0(y) \leq r < b. \quad (15)$$

For each $y \in S$, choose $r_1(y) \geq \max[r_0(y), r_0]$. (Recall (14)). Then

$$\int_r^b f(x, y) dx = \int_r^{r_1(y)} f(x, y) dx + \int_{r_1(y)}^b f(x, y) dx,$$

so (12), (15), and the triangle inequality imply that

$$\left| \int_r^b f(x, y) dx \right| < 2\epsilon, \quad y \in S, \quad r_0 \leq r < b.$$

□

In practice, we don't explicitly exhibit r_0 for each given ϵ . It suffices to obtain estimates that clearly imply its existence.

Example 4 For the improper integral of Example 2,

$$\left| \int_r^\infty f(x, y) dx \right| = \int_r^\infty |y| e^{-|y|x} = e^{-r|y|}, \quad y \neq 0.$$

If $|y| \geq \rho$, then

$$\left| \int_r^\infty f(x, y) dx \right| \leq e^{-r\rho},$$

so $\int_0^\infty f(x, y) dx$ converges uniformly on $(-\infty, \rho] \cup [\rho, \infty)$ if $\rho > 0$; however, it does not converge uniformly on any neighborhood of $y = 0$, since, for any $r > 0$, $e^{-r|y|} > \frac{1}{2}$ if $|y|$ is sufficiently small.

Definition 2 *If the improper integral*

$$\int_a^b f(x, y) dx = \lim_{r \rightarrow a+} \int_r^b f(x, y) dx$$

converges on S , it is said to converge uniformly (or be uniformly convergent) on S if, for each $\epsilon > 0$, there is an $r_0 \in (a, b]$ such that

$$\left| \int_a^b f(x, y) dx - \int_r^b f(x, y) dx \right| < \epsilon, \quad y \in S, \quad a < r \leq r_0,$$

or, equivalently,

$$\left| \int_a^r f(x, y) dx \right| < \epsilon, \quad y \in S, \quad a < r \leq r_0.$$

We leave proof of the following theorem to you (Exercise 3).

Theorem 5 (Cauchy Criterion for Uniform Convergence II) *The improper integral*

$$\int_a^b f(x, y) dx = \lim_{r \rightarrow a+} \int_r^b f(x, y) dx$$

converges uniformly on S if and only if, for each $\epsilon > 0$, there is an $r_0 \in (a, b]$ such that

$$\left| \int_{r_1}^r f(x, y) dx \right| < \epsilon, \quad y \in S, \quad a < r, r_1 \leq r_0.$$

We need one more definition, as follows.

Definition 3 *Let $f = f(x, y)$ be defined on $(a, b) \times S$, where $-\infty \leq a < b \leq \infty$. Suppose f is locally integrable on (a, b) for all $y \in S$ and let c be an arbitrary point in (a, b) . Then $\int_a^b f(x, y) dx$ is said to converge uniformly on S if $\int_a^c f(x, y) dx$ and $\int_c^b f(x, y) dx$ both converge uniformly on S .*

We leave it to you (Exercise 4) to show that this definition is independent of c ; that is, if $\int_a^c f(x, y) dx$ and $\int_c^b f(x, y) dx$ both converge uniformly on S for some $c \in (a, b)$, then they both converge uniformly on S for every $c \in (a, b)$.

We also leave it to you (Exercise 5) to show that if f is bounded on $[a, b] \times [c, d]$ and $\int_a^b f(x, y) dx$ exists as a proper integral for each $y \in [c, d]$, then it converges uniformly on $[c, d]$ according to all three Definitions 1–3.

Example 5 Consider the improper integral

$$F(y) = \int_0^\infty x^{-1/2} e^{-xy} dx,$$

which diverges if $y \leq 0$ (verify). Definition 3 applies if $y > 0$, so we consider the improper integrals

$$F_1(y) = \int_0^1 x^{-1/2} e^{-xy} dx \quad \text{and} \quad F_2(y) = \int_1^\infty x^{-1/2} e^{-xy} dx$$

separately. Moreover, we could just as well define

$$F_1(y) = \int_0^c x^{-1/2} e^{-xy} dx \quad \text{and} \quad F_2(y) = \int_c^\infty x^{-1/2} e^{-xy} dx, \quad (16)$$

where c is any positive number.

Definition 2 applies to F_1 . If $0 < r_1 < r$ and $y \geq 0$, then

$$\left| \int_r^{r_1} x^{-1/2} e^{-xy} dx \right| < \int_{r_1}^r x^{-1/2} dx < 2r^{1/2},$$

so $F_1(y)$ converges for uniformly on $[0, \infty)$.

Definition 1 applies to F_2 . Since

$$\left| \int_r^{r_1} x^{-1/2} e^{-xy} dx \right| < r^{-1/2} \int_r^\infty e^{-xy} dx = \frac{e^{-ry}}{yr^{1/2}},$$

$F_2(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$. It does not converge uniformly on $(0, \rho)$, since the change of variable $u = xy$ yields

$$\int_r^{r_1} x^{-1/2} e^{-xy} dx = y^{-1/2} \int_{ry}^{r_1 y} u^{-1/2} e^{-u} du,$$

which, for any fixed $r > 0$, can be made arbitrarily large by taking y sufficiently small and $r = 1/y$. Therefore we conclude that $F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

Note that the constant c in (16) plays no role in this argument.

Example 6 Suppose we take

$$\int_0^\infty \frac{\sin u}{u} du = \frac{\pi}{2} \quad (17)$$

as given (Exercise 31(b)). Substituting $u = xy$ with $y > 0$ yields

$$\int_0^\infty \frac{\sin xy}{x} dx = \frac{\pi}{2}, \quad y > 0. \quad (18)$$

What about uniform convergence? Since $(\sin xy)/x$ is continuous at $x = 0$, Definition 1 and Theorem 4 apply here. If $0 < r < r_1$ and $y > 0$, then

$$\int_r^{r_1} \frac{\sin xy}{x} dx = -\frac{1}{y} \left(\frac{\cos xy}{x} \Big|_r^{r_1} + \int_r^{r_1} \frac{\cos xy}{x^2} dx \right), \text{ so } \left| \int_r^{r_1} \frac{\sin xy}{x} dx \right| < \frac{3}{ry}.$$

Therefore (18) converges uniformly on $[\rho, \infty)$ if $\rho > 0$. On the other hand, from (17), there is a $\delta > 0$ such that

$$\int_{u_0}^\infty \frac{\sin u}{u} du > \frac{\pi}{4}, \quad 0 \leq u_0 < \delta.$$

This and (18) imply that

$$\int_r^\infty \frac{\sin xy}{x} dx = \int_{yr}^\infty \frac{\sin u}{u} du > \frac{\pi}{4}$$

for any $r > 0$ if $0 < y < \delta/r$. Hence, (18) does not converge uniformly on any interval $(0, \rho]$ with $\rho > 0$.

5 Absolutely Uniformly Convergent Improper Integrals

Definition 4 (Absolute Uniform Convergence I) *The improper integral*

$$\int_a^b f(x, y) dx = \lim_{r \rightarrow b-} \int_a^r f(x, y) dx$$

is said to converge absolutely uniformly on S if the improper integral

$$\int_a^b |f(x, y)| dx = \lim_{r \rightarrow b-} \int_a^r |f(x, y)| dx$$

converges uniformly on S ; that is, if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that

$$\left| \int_a^b |f(x, y)| dx - \int_a^r |f(x, y)| dx \right| < \epsilon, \quad y \in S, \quad r_0 < r < b.$$

To see that this definition makes sense, recall that if f is locally integrable on $[a, b)$ for all y in S , then so is $|f|$ (Theorem 3.4.9, p. 161). Theorem 4 with f replaced by $|f|$ implies that $\int_a^b f(x, y) dx$ converges absolutely uniformly on S if and only if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that

$$\int_r^{r_1} |f(x, y)| dx < \epsilon, \quad y \in S, \quad r_0 \leq r < r_1 < b.$$

Since

$$\left| \int_r^{r_1} f(x, y) dx \right| \leq \int_r^{r_1} |f(x, y)| dx,$$

Theorem 4 implies that if $\int_a^b f(x, y) dx$ converges absolutely uniformly on S then it converges uniformly on S .

Theorem 6 (Weierstrass's Test for Absolute Uniform Convergence I) *Suppose $M = M(x)$ is nonnegative on $[a, b)$, $\int_a^b M(x) dx < \infty$, and*

$$|f(x, y)| \leq M(x), \quad y \in S, \quad a \leq x < b. \quad (19)$$

Then $\int_a^b f(x, y) dx$ converges absolutely uniformly on S .

Proof Denote $\int_a^b M(x) dx = L < \infty$. By definition, for each $\epsilon > 0$ there is an $r_0 \in [a, b)$ such that

$$L - \epsilon < \int_a^r M(x) dx \leq L, \quad r_0 < r < b.$$

Therefore, if $r_0 < r \leq r_1$, then

$$0 \leq \int_r^{r_1} M(x) dx = \left(\int_a^{r_1} M(x) dx - L \right) - \left(\int_a^r M(x) dx - L \right) < \epsilon$$

This and (19) imply that

$$\int_r^{r_1} |f(x, y)| dx \leq \int_r^{r_1} M(x) dx < \epsilon, \quad y \in S, \quad a \leq r_0 < r < r_1 < b.$$

Now Theorem 4 implies the stated conclusion. \square

Example 7 Suppose $g = g(x, y)$ is locally integrable on $[0, \infty)$ for all $y \in S$ and, for some $a_0 \geq 0$, there are constants K and p_0 such that

$$|g(x, y)| \leq K e^{p_0 x}, \quad y \in S, \quad x \geq a_0.$$

If $p > p_0$ and $r \geq a_0$, then

$$\begin{aligned} \int_r^\infty e^{-px} |g(x, y)| dx &= \int_r^\infty e^{-(p-p_0)x} e^{-p_0 x} |g(x, y)| dx \\ &\leq K \int_r^\infty e^{-(p-p_0)x} dx = \frac{K e^{-(p-p_0)r}}{p - p_0}, \end{aligned}$$

so $\int_0^\infty e^{-px} g(x, y) dx$ converges absolutely on S . For example, since

$$|x^\alpha \sin xy| < e^{p_0 x} \quad \text{and} \quad |x^\alpha \cos xy| < e^{p_0 x}$$

for x sufficiently large if $p_0 > 0$, Theorem 4 implies that $\int_0^\infty e^{-px} x^\alpha \sin xy dx$ and $\int_0^\infty e^{-px} x^\alpha \cos xy dx$ converge absolutely uniformly on $(-\infty, \infty)$ if $p > 0$ and $\alpha \geq 0$. As a matter of fact, $\int_0^\infty e^{-px} x^\alpha \sin xy dx$ converges absolutely on $(-\infty, \infty)$ if $p > 0$ and $\alpha > -1$. (Why?)

Definition 5 (Absolute Uniform Convergence II) *The improper integral*

$$\int_a^b f(x, y) dx = \lim_{r \rightarrow a^+} \int_r^b f(x, y) dx$$

is said to converge absolutely uniformly on S if the improper integral

$$\int_a^b |f(x, y)| dx = \lim_{r \rightarrow a^+} \int_r^b |f(x, y)| dx$$

converges uniformly on S ; that is, if, for each $\epsilon > 0$, there is an $r_0 \in (a, b]$ such that

$$\left| \int_a^b |f(x, y)| dx - \int_r^b |f(x, y)| dx \right| < \epsilon, \quad y \in S, \quad a < r < r_0 \leq b.$$

We leave it to you (Exercise 7) to prove the following theorem.

Theorem 7 (Weierstrass's Test for Absolute Uniform Convergence II) *Suppose $M = M(x)$ is nonnegative on $(a, b]$, $\int_a^b M(x) dx < \infty$, and*

$$|f(x, y)| \leq M(x), \quad y \in S, \quad x \in (a, b].$$

Then $\int_a^b f(x, y) dx$ converges absolutely uniformly on S .

Example 8 If $g = g(x, y)$ is locally integrable on $(0, 1]$ for all $y \in S$ and

$$|g(x, y)| \leq Ax^{-\beta}, \quad 0 < x \leq x_0,$$

for each $y \in S$, then

$$\int_0^1 x^\alpha g(x, y) dx$$

converges absolutely uniformly on S if $\alpha > \beta - 1$. To see this, note that if $0 < r < r_1 \leq x_0$, then

$$\int_{r_1}^r x^\alpha |g(x, y)| dx \leq A \int_{r_1}^r x^{\alpha-\beta} dx = \frac{Ax^{\alpha-\beta+1}}{\alpha-\beta+1} \Big|_{r_1}^r < \frac{Ar^{\alpha-\beta+1}}{\alpha-\beta+1}.$$

Applying this with $\beta = 0$ shows that

$$F(y) = \int_0^1 x^\alpha \cos xy dx$$

converges absolutely uniformly on $(-\infty, \infty)$ if $\alpha > -1$ and

$$G(y) = \int_0^1 x^\alpha \sin xy dx$$

converges absolutely uniformly on $(-\infty, \infty)$ if $\alpha > -2$.

By recalling Theorem 4.4.15 (p. 246), you can see why we associate Theorems 6 and 7 with Weierstrass.

6 Dirichlet's Tests

Weierstrass's test is useful and important, but it has a basic shortcoming: it applies only to absolutely uniformly convergent improper integrals. The next theorem applies in some cases where $\int_a^b f(x, y) dx$ converges uniformly on S , but $\int_a^b |f(x, y)| dx$ does not.

Theorem 8 (Dirichlet's Test for Uniform Convergence I) If g , g_x , and h are continuous on $[a, b) \times S$, then

$$\int_a^b g(x, y)h(x, y) dx$$

converges uniformly on S if the following conditions are satisfied:

- (a) $\lim_{x \rightarrow b-} \left\{ \sup_{y \in S} |g(x, y)| \right\} = 0$;
- (b) There is a constant M such that

$$\sup_{y \in S} \left| \int_a^x h(u, y) du \right| < M, \quad a \leq x < b;$$

(c) $\int_a^b |g_x(x, y)| dx$ converges uniformly on S .

Proof If

$$H(x, y) = \int_a^x h(u, y) du, \quad (20)$$

then integration by parts yields

$$\begin{aligned} \int_r^{r_1} g(x, y) h(x, y) dx &= \int_r^{r_1} g(x, y) H_x(x, y) dx \\ &= g(r_1, y) H(r_1, y) - g(r, y) H(r, y) \\ &\quad - \int_r^{r_1} g_x(x, y) H(x, y) dx. \end{aligned} \quad (21)$$

Since assumption (b) and (20) imply that $|H(x, y)| \leq M$, $(x, y) \in (a, b] \times S$, Eqn. (21) implies that

$$\left| \int_r^{r_1} g(x, y) h(x, y) dx \right| < M \left(2 \sup_{x \geq r} |g(x, y)| + \int_r^{r_1} |g_x(x, y)| dx \right) \quad (22)$$

on $[r, r_1] \times S$.

Now suppose $\epsilon > 0$. From assumption (a), there is an $r_0 \in [a, b)$ such that $|g(x, y)| < \epsilon$ on S if $r_0 \leq x < b$. From assumption (c) and Theorem 6, there is an $s_0 \in [a, b)$ such that

$$\int_r^{r_1} |g_x(x, y)| dx < \epsilon, \quad y \in S, \quad s_0 < r < r_1 < b.$$

Therefore (22) implies that

$$\left| \int_r^{r_1} g(x, y) h(x, y) dx \right| < 3M\epsilon, \quad y \in S, \quad \max(r_0, s_0) < r < r_1 < b.$$

Now Theorem 4 implies the stated conclusion. \square

The statement of this theorem is complicated, but applying it isn't; just look for a factorization $f = gh$, where h has a bounded antiderivative on $[a, b)$ and g is "small" near b . Then integrate by parts and hope that something nice happens. A similar comment applies to Theorem 9, which follows.

Example 9 Let

$$I(y) = \int_0^\infty \frac{\cos xy}{x + y} dx, \quad y > 0.$$

The obvious inequality

$$\left| \frac{\cos xy}{x + y} \right| \leq \frac{1}{x + y}$$

is useless here, since

$$\int_0^\infty \frac{dx}{x + y} = \infty.$$

However, integration by parts yields

$$\begin{aligned}\int_r^{r_1} \frac{\cos xy}{x+y} dx &= \left. \frac{\sin xy}{y(x+y)} \right|_r^{r_1} + \int_r^{r_1} \frac{\sin xy}{y(x+y)^2} dx \\ &= \frac{\sin r_1 y}{y(r_1+y)} - \frac{\sin ry}{y(r+y)} + \int_r^{r_1} \frac{\sin xy}{y(x+y)^2} dx.\end{aligned}$$

Therefore, if $0 < r < r_1$, then

$$\left| \int_r^{r_1} \frac{\cos xy}{x+y} dx \right| < \frac{1}{y} \left(\frac{2}{r+y} + \int_r^\infty \frac{1}{(x+y)^2} \right) \leq \frac{3}{y(r+y)^2} \leq \frac{3}{\rho(r+\rho)}$$

if $y \geq \rho > 0$. Now Theorem 4 implies that $I(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

We leave the proof of the following theorem to you (Exercise 10).

Theorem 9 (Dirichlet's Test for Uniform Convergence II) *If g , g_x , and h are continuous on $(a, b] \times S$, then*

$$\int_a^b g(x, y)h(x, y) dx$$

converges uniformly on S if the following conditions are satisfied:

- (a) $\lim_{x \rightarrow a+} \left\{ \sup_{y \in S} |g(x, y)| \right\} = 0$;
- (b) *There is a constant M such that*

$$\sup_{y \in S} \left| \int_x^b h(u, y) du \right| \leq M, \quad a < x \leq b;$$

- (c) $\int_a^b |g_x(x, y)| dx$ *converges uniformly on S .*

By recalling Theorems 3.4.10 (p. 163), 4.3.20 (p. 217), and 4.4.16 (p. 248), you can see why we associate Theorems 8 and 9 with Dirichlet.

7 Consequences of uniform convergence

Theorem 10 *If $f = f(x, y)$ is continuous on either $[a, b] \times [c, d]$ or $(a, b) \times [c, d]$ and*

$$F(y) = \int_a^b f(x, y) dx \tag{23}$$

converges uniformly on $[c, d]$, then F is continuous on $[c, d]$. Moreover,

$$\int_c^d \left(\int_a^b f(x, y) dx \right) dy = \int_a^b \left(\int_c^d f(x, y) dy \right) dx. \tag{24}$$

Proof We will assume that f is continuous on $(a, b] \times [c, d]$. You can consider the other case (Exercise 14).

We will first show that F in (23) is continuous on $[c, d]$. Since F converges uniformly on $[c, d]$, Definition 1 (specifically, (11)) implies that if $\epsilon > 0$, there is an $r \in [a, b]$ such that

$$\left| \int_r^b f(x, y) dx \right| < \epsilon, \quad c \leq y \leq d.$$

Therefore, if $c \leq y, y_0 \leq d$, then

$$\begin{aligned} |F(y) - F(y_0)| &= \left| \int_a^b f(x, y) dx - \int_a^b f(x, y_0) dx \right| \\ &\leq \left| \int_a^r [f(x, y) - f(x, y_0)] dx \right| + \left| \int_r^b f(x, y) dx \right| \\ &\quad + \left| \int_r^b f(x, y_0) dx \right|, \end{aligned}$$

so

$$|F(y) - F(y_0)| \leq \int_a^r |f(x, y) - f(x, y_0)| dx + 2\epsilon. \quad (25)$$

Since f is uniformly continuous on the compact set $[a, r] \times [c, d]$ (Corollary 5.2.14, p. 314), there is a $\delta > 0$ such that

$$|f(x, y) - f(x, y_0)| < \epsilon$$

if (x, y) and (x, y_0) are in $[a, r] \times [c, d]$ and $|y - y_0| < \delta$. This and (25) imply that

$$|F(y) - F(y_0)| < (r - a)\epsilon + 2\epsilon < (b - a + 2)\epsilon$$

if y and y_0 are in $[c, d]$ and $|y - y_0| < \delta$. Therefore F is continuous on $[c, d]$, so the integral on left side of (24) exists. Denote

$$I = \int_c^d \left(\int_a^b f(x, y) dx \right) dy. \quad (26)$$

We will show that the improper integral on the right side of (24) converges to I . To this end, denote

$$I(r) = \int_a^r \left(\int_c^d f(x, y) dy \right) dx.$$

Since we can reverse the order of integration of the continuous function f over the rectangle $[a, r] \times [c, d]$ (Corollary 7.2.2, p. 466),

$$I(r) = \int_c^d \left(\int_a^r f(x, y) dx \right) dy.$$

From this and (26),

$$I - I(r) = \int_c^d \left(\int_r^b f(x, y) dx \right) dy.$$

Now suppose $\epsilon > 0$. Since $\int_a^b f(x, y) dx$ converges uniformly on $[c, d]$, there is an $r_0 \in (a, b]$ such that

$$\left| \int_r^b f(x, y) dx \right| < \epsilon, \quad r_0 < r < b,$$

so $|I - I(r)| < (d - c)\epsilon$ if $r_0 < r < b$. Hence,

$$\lim_{r \rightarrow b-} \int_a^r \left(\int_c^d f(x, y) dy \right) dx = \int_c^d \left(\int_a^b f(x, y) dx \right) dy,$$

which completes the proof of (24). \square

Example 10 It is straightforward to verify that

$$\int_0^\infty e^{-xy} dx = \frac{1}{y}, \quad y > 0,$$

and the convergence is uniform on $[\rho, \infty)$ if $\rho > 0$. Therefore Theorem 10 implies that if $0 < y_1 < y_2$, then

$$\begin{aligned} \int_{y_1}^{y_2} \frac{dy}{y} &= \int_{y_1}^{y_2} \left(\int_0^\infty e^{-xy} dx \right) dy = \int_0^\infty \left(\int_{y_1}^{y_2} e^{-xy} dy \right) dx \\ &= \int_0^\infty \frac{e^{-xy_1} - e^{-xy_2}}{x} dx. \end{aligned}$$

Since

$$\int_{y_1}^{y_2} \frac{dy}{y} = \log \frac{y_2}{y_1}, \quad y_2 \geq y_1 > 0,$$

it follows that

$$\int_0^\infty \frac{e^{-xy_1} - e^{-xy_2}}{x} dx = \log \frac{y_2}{y_1}, \quad y_2 \geq y_1 > 0.$$

Example 11 From Example 6,

$$\int_0^\infty \frac{\sin xy}{x} dx = \frac{\pi}{2}, \quad y > 0,$$

and the convergence is uniform on $[\rho, \infty)$ if $\rho > 0$. Therefore, Theorem 10 implies that if $0 < y_1 < y_2$, then

$$\begin{aligned} \frac{\pi}{2}(y_2 - y_1) &= \int_{y_1}^{y_2} \left(\int_0^\infty \frac{\sin xy}{x} dx \right) dy = \int_0^\infty \left(\int_{y_1}^{y_2} \frac{\sin xy}{x} dy \right) dx \\ &= \int_0^\infty \frac{\cos xy_1 - \cos xy_2}{x^2} dx. \end{aligned} \tag{27}$$

The last integral converges uniformly on $(-\infty, \infty)$ (Exercise 10(h)), and is therefore continuous with respect to y_1 on $(-\infty, \infty)$, by Theorem 10; in particular, we can let $y_1 \rightarrow 0+$ in (27) and replace y_2 by y to obtain

$$\int_0^\infty \frac{1 - \cos xy}{x^2} dx = \frac{\pi y}{2}, \quad y \geq 0.$$

The next theorem is analogous to Theorem 4.4.20 (p. 252).

Theorem 11 *Let f and f_y be continuous on either $[a, b] \times [c, d]$ or $(a, b) \times [c, d]$. Suppose that the improper integral*

$$F(y) = \int_a^b f(x, y) dx$$

converges for some $y_0 \in [c, d]$ and

$$G(y) = \int_a^b f_y(x, y) dx$$

converges uniformly on $[c, d]$. Then F converges uniformly on $[c, d]$ and is given explicitly by

$$F(y) = F(y_0) + \int_{y_0}^y G(t) dt, \quad c \leq y \leq d.$$

Moreover, F is continuously differentiable on $[c, d]$; specifically,

$$F'(y) = G(y), \quad c \leq y \leq d, \tag{28}$$

where $F'(c)$ and $f_y(x, c)$ are derivatives from the right, and $F'(d)$ and $f_y(x, d)$ are derivatives from the left.

Proof We will assume that f and f_y are continuous on $[a, b] \times [c, d]$. You can consider the other case (Exercise 15).

Let

$$F_r(y) = \int_a^r f(x, y) dx, \quad a \leq r < b, \quad c \leq y \leq d.$$

Since f and f_y are continuous on $[a, r] \times [c, d]$, Theorem 1 implies that

$$F'_r(y) = \int_a^r f_y(x, y) dx, \quad c \leq y \leq d.$$

Then

$$\begin{aligned} F_r(y) &= F_r(y_0) + \int_{y_0}^y \left(\int_a^r f_y(x, t) dx \right) dt \\ &= F(y_0) + \int_{y_0}^y G(t) dt \\ &\quad + (F_r(y_0) - F(y_0)) - \int_{y_0}^y \left(\int_r^b f_y(x, t) dx \right) dt, \quad c \leq y \leq d. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| F_r(y) - F(y_0) - \int_{y_0}^y G(t) dt \right| &\leq |F_r(y_0) - F(y_0)| \\ &\quad + \left| \int_{y_0}^y \int_r^b f_y(x, t) dx \right| dt. \end{aligned} \quad (29)$$

Now suppose $\epsilon > 0$. Since we have assumed that $\lim_{r \rightarrow b-} F_r(y_0) = F(y_0)$ exists, there is an r_0 in (a, b) such that

$$|F_r(y_0) - F(y_0)| < \epsilon, \quad r_0 < r < b.$$

Since we have assumed that $G(y)$ converges for $y \in [c, d]$, there is an $r_1 \in [a, b)$ such that

$$\left| \int_r^b f_y(x, t) dx \right| < \epsilon, \quad t \in [c, d], \quad r_1 \leq r < b.$$

Therefore, (29) yields

$$\left| F_r(y) - F(y_0) - \int_{y_0}^y G(t) dt \right| < \epsilon(1 + |y - y_0|) \leq \epsilon(1 + d - c)$$

if $\max(r_0, r_1) \leq r < b$ and $t \in [c, d]$. Therefore $F(y)$ converges uniformly on $[c, d]$ and

$$F(y) = F(y_0) + \int_{y_0}^y G(t) dt, \quad c \leq y \leq d.$$

Since G is continuous on $[c, d]$ by Theorem 10, (28) follows from differentiating this (Theorem 3.3.11, p. 141). \square

Example 12 Let

$$I(y) = \int_0^\infty e^{-yx^2} dx, \quad y > 0.$$

Since

$$\int_0^r e^{-yx^2} dx = \frac{1}{\sqrt{y}} \int_0^{r\sqrt{y}} e^{-t^2} dt,$$

it follows that

$$I(y) = \frac{1}{\sqrt{y}} \int_0^\infty e^{-t^2} dt,$$

and the convergence is uniform on $[\rho, \infty)$ if $\rho > 0$ (Exercise 8(i)). To evaluate the last integral, denote $J(\rho) = \int_0^\rho e^{-t^2} dt$; then

$$J^2(\rho) = \left(\int_0^\rho e^{-u^2} du \right) \left(\int_0^\rho e^{-v^2} dv \right) = \int_0^\rho \int_0^\rho e^{-(u^2+v^2)} du dv.$$

Transforming to polar coordinates $r = r \cos \theta$, $v = r \sin \theta$ yields

$$J^2(\rho) = \int_0^{\pi/2} \int_0^\rho r e^{-r^2} dr d\theta = \frac{\pi(1 - e^{-\rho^2})}{4}, \quad \text{so} \quad J(\rho) = \frac{\sqrt{\pi(1 - e^{-\rho^2})}}{2}.$$

Therefore

$$\int_0^\infty e^{-t^2} dt = \lim_{\rho \rightarrow \infty} J(\rho) = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad \int_0^\infty e^{-yx^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{y}}, \quad y > 0.$$

Differentiating this n times with respect to y yields

$$\int_0^\infty x^{2n} e^{-yx^2} dx = \frac{1 \cdot 3 \cdots (2n-1) \sqrt{\pi}}{2^n y^{n+1/2}} \quad y > 0, \quad n = 1, 2, 3, \dots,$$

where Theorem 11 justifies the differentiation for every n , since all these integrals converge uniformly on $[\rho, \infty)$ if $\rho > 0$ (Exercise 8(i)).

Some advice for applying this theorem: Be sure to check first that $F(y_0) = \int_a^b f(x, y_0) dx$ converges for at least one value of y . If so, differentiate $\int_a^b f(x, y) dx$ formally to obtain $\int_a^b f_y(x, y) dx$. Then $F'(y) = \int_a^b f_y(x, y) dx$ if y is in some interval on which this improper integral converges uniformly.

8 Applications to Laplace transforms

The *Laplace transform* of a function f locally integrable on $[0, \infty)$ is

$$F(s) = \int_0^\infty e^{-sx} f(x) dx$$

for all s such that integral converges. Laplace transforms are widely applied in mathematics, particularly in solving differential equations.

We leave it to you to prove the following theorem (Exercise 26).

Theorem 12 Suppose f is locally integrable on $[0, \infty)$ and $|f(x)| \leq M e^{s_0 x}$ for sufficiently large x . Then the Laplace transform of F converges uniformly on $[s_1, \infty)$ if $s_1 > s_0$.

Theorem 13 If f is continuous on $[0, \infty)$ and $H(x) = \int_0^\infty e^{-s_0 u} f(u) du$ is bounded on $[0, \infty)$, then the Laplace transform of f converges uniformly on $[s_1, \infty)$ if $s_1 > s_0$.

Proof If $0 \leq r \leq r_1$,

$$\int_r^{r_1} e^{-sx} f(x) dx = \int_r^{r_1} e^{-(s-s_0)x} e^{-s_0 x} f(x) dx = \int_r^{r_1} e^{-(s-s_0)t} H'(x) dt.$$

Integration by parts yields

$$\int_r^{r_1} e^{-sx} f(x) dx = e^{-(s-s_0)x} H(x) \Big|_r^{r_1} + (s-s_0) \int_r^{r_1} e^{-(s-s_0)x} H(x) dx.$$

Therefore, if $|H(x)| \leq M$, then

$$\begin{aligned} \left| \int_r^{r_1} e^{-sx} f(x) dx \right| &\leq M \left| e^{-(s-s_0)r_1} + e^{-(s-s_0)r} + (s-s_0) \int_r^{r_1} e^{-(s-s_0)x} dx \right| \\ &\leq 3Me^{-(s-s_0)r} \leq 3Me^{-(s_1-s_0)r}, \quad s \geq s_1. \end{aligned}$$

Now Theorem 4 implies that $F(s)$ converges uniformly on $[s_1, \infty)$.

The following theorem draws a considerably stronger conclusion from the same assumptions.

Theorem 14 *If f is continuous on $[0, \infty)$ and*

$$H(x) = \int_0^x e^{-s_0 u} f(u) du$$

is bounded on $[0, \infty)$, then the Laplace transform of f is infinitely differentiable on (s_0, ∞) , with

$$F^{(n)}(s) = (-1)^n \int_0^\infty e^{-sx} x^n f(x) dx; \quad (30)$$

that is, the n -th derivative of the Laplace transform of $f(x)$ is the Laplace transform of $(-1)^n x^n f(x)$.

Proof First we will show that the integrals

$$I_n(s) = \int_0^\infty e^{-sx} x^n f(x) dx, \quad n = 0, 1, 2, \dots$$

all converge uniformly on $[s_1, \infty)$ if $s_1 > s_0$. If $0 < r < r_1$, then

$$\int_r^{r_1} e^{-sx} x^n f(x) dx = \int_r^{r_1} e^{-(s-s_0)x} e^{-s_0 x} x^n f(x) dx = \int_r^{r_1} e^{-(s-s_0)x} x^n H'(x) dx.$$

Integrating by parts yields

$$\begin{aligned} \int_r^{r_1} e^{-sx} x^n f(x) dx &= r_1^n e^{-(s-s_0)r_1} H(r) - r^n e^{-(s-s_0)r} H(r) \\ &\quad - \int_r^{r_1} H(x) \left(e^{-(s-s_0)x} x^n \right)' dx, \end{aligned}$$

where $'$ indicates differentiation with respect to x . Therefore, if $|H(x)| \leq M \leq \infty$ on $[0, \infty)$, then

$$\left| \int_r^{r_1} e^{-sx} x^n f(x) dx \right| \leq M \left(e^{-(s-s_0)r} r^n + e^{-(s-s_0)r_1} r_1^n + \int_r^{r_1} |(e^{-(s-s_0)x} x^n)'| dx \right).$$

Therefore, since $e^{-(s-s_0)r} r^n$ decreases monotonically on (n, ∞) if $s > s_0$ (check!),

$$\left| \int_r^{r_1} e^{-sx} x^n f(x) dx \right| < 3Me^{-(s-s_0)r} r^n, \quad n < r < r_1,$$

so Theorem 4 implies that $I_n(s)$ converges uniformly $[s_1, \infty)$ if $s_1 > s_0$. Now Theorem 11 implies that $F_{n+1} = -F'_n$, and an easy induction proof yields (30) (Exercise 25). \square

Example 13 Here we apply Theorem 12 with $f(x) = \cos ax$ ($a \neq 0$) and $s_0 = 0$. Since

$$\int_0^x \cos au \, du = \frac{\sin ax}{a}$$

is bounded on $(0, \infty)$, Theorem 12 implies that

$$F(s) = \int_0^\infty e^{-sx} \cos ax \, dx$$

converges and

$$F^{(n)}(s) = (-1)^n \int_0^\infty e^{-sx} x^n \cos ax \, dx, \quad s > 0. \quad (31)$$

(Note that this is also true if $a = 0$.) Elementary integration yields

$$F(s) = \frac{s}{s^2 + a^2}.$$

Hence, from (31),

$$\int_0^\infty e^{-sx} x^n \cos ax \, dx = (-1)^n \frac{d^n}{ds^n} \frac{s}{s^2 + a^2}, \quad n = 0, 1, \dots$$

9 Exercises

1. Suppose g and h are differentiable on $[a, b]$, with

$$a \leq g(y) \leq b \quad \text{and} \quad a \leq h(y) \leq b, \quad c \leq y \leq d.$$

Let f and f_y be continuous on $[a, b] \times [c, d]$. Derive *Liebniz's rule*:

$$\begin{aligned} \frac{d}{dy} \int_{g(y)}^{h(y)} f(x, y) dx &= f(h(y), y)h'(y) - f(g(y), y)g'(y) \\ &\quad + \int_{g(y)}^{h(y)} f_y(x, y) dx. \end{aligned}$$

(Hint: Define $H(y, u, v) = \int_u^v f(x, y) dx$ and use the chain rule.)

2. Adapt the proof of Theorem 2 to prove Theorem 3.
3. Adapt the proof of Theorem 4 to prove Theorem 5.
4. Show that Definition 3 is independent of c ; that is, if $\int_a^c f(x, y) dx$ and $\int_c^b f(x, y) dx$ both converge uniformly on S for some $c \in (a, b)$, then they both converge uniformly on S and every $c \in (a, b)$.
5. (a) Show that if f is bounded on $[a, b] \times [c, d]$ and $\int_a^b f(x, y) dx$ exists as a proper integral for each $y \in [c, d]$, then it converges uniformly on $[c, d]$ according to all of Definition 1–3.
 (b) Give an example to show that the boundedness of f is essential in (a).
6. Working directly from Definition 1, discuss uniform convergence of the following integrals:
- (a) $\int_0^\infty \frac{dx}{1 + y^2 x^2} dx$ (b) $\int_0^\infty e^{-xy} x^2 dx$
- (c) $\int_0^\infty x^{2n} e^{-yx^2} dx$ (d) $\int_0^\infty \sin xy^2 dx$
- (e) $\int_0^\infty (3y^2 - 2xy) e^{-y^2 x} dx$ (f) $\int_0^\infty (2xy - y^2 x^2) e^{-xy} dx$
7. Adapt the proof of Theorem 6 to prove Theorem 7.
8. Use Weierstrass's test to show that the integral converges uniformly on S :
- (a) $\int_0^\infty e^{-xy} \sin x dx, \quad S = [\rho, \infty), \quad \rho > 0$
- (b) $\int_0^\infty \frac{\sin x}{x^y} dx, \quad S = [c, d], \quad 1 < c < d < 2$

- (c) $\int_1^\infty e^{-px} \frac{\sin xy}{x} dx, \quad p > 0, \quad S = (-\infty, \infty)$
- (d) $\int_0^1 \frac{e^{xy}}{(1-x)^y} dx, \quad S = (-\infty, b), \quad b < 1$
- (e) $\int_{-\infty}^\infty \frac{\cos xy}{1+x^2y^2} dx, \quad S = (-\infty, -\rho] \cup [\rho, \infty), \quad \rho > 0.$
- (f) $\int_1^\infty e^{-x/y} dx, \quad S = [\rho, \infty), \quad \rho > 0$
- (g) $\int_{-\infty}^\infty e^{xy} e^{-x^2} dx, \quad S = [-\rho, \rho], \quad \rho > 0$
- (h) $\int_0^\infty \frac{\cos xy - \cos ax}{x^2} dx, \quad S = (-\infty, \infty)$
- (i) $\int_0^\infty x^{2n} e^{-yx^2} dx, \quad S = [\rho, \infty), \quad \rho > 0, \quad n = 0, 1, 2, \dots$

9. (a) Show that

$$\Gamma(y) = \int_0^\infty x^{y-1} e^{-x} dx$$

converges if $y > 0$, and uniformly on $[c, d]$ if $0 < c < d < \infty$.

(b) Use integration by parts to show that

$$\Gamma(y) = \frac{\Gamma(y+1)}{y}, \quad y \geq 0,$$

and then show by induction that

$$\Gamma(y) = \frac{\Gamma(y+n)}{y(y+1)\cdots(y+n-1)}, \quad y > 0, \quad n = 1, 2, 3, \dots$$

How can this be used to define $\Gamma(y)$ in a natural way for all $y \neq 0, -1, -2, \dots$? (This function is called the *gamma function*.)

(c) Show that $\Gamma(n+1) = n!$ if n is a positive integer.

(d) Show that

$$\int_0^\infty e^{-st} t^\alpha dt = s^{-\alpha-1} \Gamma(\alpha+1), \quad \alpha > -1, \quad s > 0.$$

10. Show that Theorem 8 remains valid with assumption (c) replaced by the assumption that $|g_x(x, y)|$ is monotonic with respect to x for all $y \in S$.

11. Adapt the proof of Theorem 8 to prove Theorem 9.

12. Use Dirichlet's test to show that the following integrals converge uniformly on $S = [\rho, \infty)$ if $\rho > 0$:

$$\begin{aligned} \text{(a)} \int_1^\infty \frac{\sin xy}{x^y} dx & \quad \text{(b)} \int_2^\infty \frac{\sin xy}{\log x} dx \\ \text{(c)} \int_0^\infty \frac{\cos xy}{x+y^2} dx & \quad \text{(d)} \int_1^\infty \frac{\sin xy}{1+xy} dx \end{aligned}$$

13. Suppose g , g_x and h are continuous on $[a, b) \times S$, and denote $H(x, y) = \int_a^x h(u, y) du$, $a \leq x < b$. Suppose also that

$$\lim_{x \rightarrow b-} \left\{ \sup_{y \in S} |g(x, y)H(x, y)| \right\} = 0 \quad \text{and} \quad \int_a^b g_x(x, y)H(x, y) dx$$

converges uniformly on S . Show that $\int_a^b g(x, y)h(x, y) dx$ converges uniformly on S .

14. Prove Theorem 10 for the case where $f = f(x, y)$ is continuous on $(a, b] \times [c, d]$.

15. Prove Theorem 11 for the case where $f = f(x, y)$ is continuous on $(a, b] \times [c, d]$.

16. Show that

$$C(y) = \int_{-\infty}^\infty f(x) \cos xy \, dx \quad \text{and} \quad S(y) = \int_{-\infty}^\infty f(x) \sin xy \, dx$$

are continuous on $(-\infty, \infty)$ if

$$\int_{-\infty}^\infty |f(x)| \, dx < \infty.$$

17. Suppose f is continuously differentiable on $[a, \infty)$, $\lim_{x \rightarrow \infty} f(x) = 0$, and

$$\int_a^\infty |f'(x)| \, dx < \infty.$$

Show that the functions

$$C(y) = \int_a^\infty f(x) \cos xy \, dx \quad \text{and} \quad S(y) = \int_a^\infty f(x) \sin xy \, dx$$

are continuous for all $y \neq 0$. Give an example showing that they need not be continuous at $y = 0$.

18. Evaluate $F(y)$ and use Theorem 11 to evaluate I :

$$\text{(a)} \quad F(y) = \int_0^\infty \frac{dx}{1+y^2x^2}, \quad y \neq 0; \quad I = \int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x} dx, \\ a, b > 0$$

$$(b) \quad F(y) = \int_0^\infty x^y dx, \quad y > -1; \quad I = \int_0^\infty \frac{x^a - x^b}{\log x} dx, \quad a, b > -1$$

$$(c) \quad F(y) = \int_0^\infty e^{-xy} \cos x dx, \quad y > 0$$

$$I = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \cos x dx, \quad a, b > 0$$

$$(d) \quad F(y) = \int_0^\infty e^{-xy} \sin x dx, \quad y > 0$$

$$I = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \sin x dx, \quad a, b > 0$$

$$(e) \quad F(y) = \int_0^\infty e^{-x} \sin xy dx; \quad I = \int_0^\infty e^{-x} \frac{1 - \cos ax}{x} dx$$

$$(f) \quad F(y) = \int_0^\infty e^{-x} \cos xy dx; \quad I = \int_0^\infty e^{-x} \frac{\sin ax}{x} dx$$

19. Use Theorem 11 to evaluate:

$$(a) \quad \int_0^1 (\log x)^n x^y dx, \quad y > -1, \quad n = 0, 1, 2, \dots$$

$$(b) \quad \int_0^\infty \frac{dx}{(x^2 + y)^{n+1}}, \quad y > 0, \quad n = 0, 1, 2, \dots$$

$$(c) \quad \int_0^\infty x^{2n+1} e^{-yx^2} dx, \quad y > 0, \quad n = 0, 1, 2, \dots$$

$$(d) \quad \int_0^\infty xy^x dx, \quad 0 < y < 1.$$

20. (a) Use Theorem 11 and integration by parts to show that

$$F(y) = \int_0^\infty e^{-x^2} \cos 2xy dx$$

satisfies

$$F' + 2yF = 0.$$

(b) Use part (a) to show that

$$F(y) = \frac{\sqrt{\pi}}{2} e^{-y^2}.$$

21. Show that

$$\int_0^\infty e^{-x^2} \sin 2xy dx = e^{-y^2} \int_0^y e^{u^2} du.$$

(Hint: See Exercise 20.)

22. State a condition implying that

$$C(y) = \int_a^\infty f(x) \cos xy dx \quad \text{and} \quad S(y) = \int_a^\infty f(x) \sin xy dx$$

are n times differentiable on for all $y \neq 0$. (Your condition should imply the hypotheses of Exercise 16.)

23. Suppose f is continuously differentiable on $[a, \infty)$,

$$\int_a^\infty |(x^k f(x))'| dx < \infty, \quad 0 \leq k \leq n,$$

and $\lim_{x \rightarrow \infty} x^n f(x) = 0$. Show that if

$$C(y) = \int_a^\infty f(x) \cos xy \, dx \quad \text{and} \quad S(y) = \int_a^\infty f(x) \sin xy \, dx,$$

then

$$C^{(k)}(y) = \int_a^\infty x^k f(x) \cos xy \, dx \quad \text{and} \quad S^{(k)}(y) = \int_a^\infty x^k f(x) \sin xy \, dx,$$

$$0 \leq k \leq n.$$

24. Differentiating

$$F(y) = \int_1^\infty \cos \frac{y}{x} \, dx$$

under the integral sign yields

$$- \int_1^\infty \frac{1}{x} \sin \frac{y}{x} \, dx,$$

which converges uniformly on any finite interval. (Why?) Does this imply that F is differentiable for all y ?

25. Show that Theorem 11 and induction imply Eq. (30).

26. Prove Theorem 12.

27. Show that if $F(s) = \int_0^\infty e^{-sx} f(x) \, dx$ converges for $s = s_0$, then it converges uniformly on $[s_0, \infty)$. (What's the difference between this and Theorem 13?)

28. Prove: If f is continuous on $[0, \infty)$ and $\int_0^\infty e^{-s_0 x} f(x) \, dx$ converges, then

$$\lim_{s \rightarrow s_0 +} \int_0^\infty e^{-sx} f(x) \, dx = \int_0^\infty e^{-s_0 x} f(x) \, dx.$$

(Hint: See the proof of Theorem 4.5.12, p. 273.)

29. Under the assumptions of Exercise 28, show that

$$\lim_{s \rightarrow s_0 +} \int_r^\infty e^{-sx} f(x) \, dx = \int_r^\infty e^{-s_0 x} f(x) \, dx, \quad r > 0.$$

30. Suppose f is continuous on $[0, \infty)$ and

$$F(s) = \int_0^{\infty} e^{-sx} f(x) dx$$

converges for $s = s_0$. Show that $\lim_{s \rightarrow \infty} F(s) = 0$. (Hint: Integrate by parts.)

31. (a) Starting from the result of Exercise 18(d), let $b \rightarrow \infty$ and invoke Exercise 30 to evaluate

$$\int_0^{\infty} e^{-ax} \frac{\sin x}{x} dx, \quad a > 0.$$

- (b) Use (a) and Exercise 28 to show that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

32. (a) Suppose f is continuously differentiable on $[0, \infty)$ and

$$|f(x)| \leq M e^{s_0 x}, \quad 0 \leq x \leq \infty.$$

Show that

$$G(s) = \int_0^{\infty} e^{-sx} f'(x) dx$$

converges uniformly on $[s_1, \infty)$ if $s_1 > s_0$. (Hint: Integrate by parts.)

- (b) Show from part (a) that

$$G(s) = \int_0^{\infty} e^{-sx} x e^{x^2} \sin e^{x^2} dx$$

converges uniformly on $[\rho, \infty)$ if $\rho > 0$. (Notice that this does not follow from Theorem 6 or 8.)

33. Suppose f is continuous on $[0, \infty)$,

$$\lim_{x \rightarrow 0+} \frac{f(x)}{x}$$

exists, and

$$F(s) = \int_0^{\infty} e^{-sx} f(x) dx$$

converges for $s = s_0$. Show that

$$\int_{s_0}^{\infty} F(u) du = \int_0^{\infty} e^{-s_0 x} \frac{f(x)}{x} dx.$$

10 Answers to selected exercises

5. (b) If $f(x, y) = 1/y$ for $y \neq 0$ and $f(x, 0) = 1$, then $\int_a^b f(x, y) dx$ does not converge uniformly on $[0, d]$ for any $d > 0$.

6. (a), (d), and (e) converge uniformly on $(-\infty, \rho] \cup [\rho, \infty)$ if $\rho > 0$; **(b), (c), and (f)** converge uniformly on $[\rho, \infty)$ if $\rho > 0$.

17. Let $C(y) = \int_1^\infty \frac{\cos xy}{x} dx$ and $S(y) = \int_1^\infty \frac{\sin xy}{x} dx$. Then $C(0) = \infty$ and $S(0) = 0$, while $S(y) = \pi/2$ if $y \neq 0$.

18. (a) $F(y) = \frac{\pi}{2|y|}$; $I = \frac{\pi}{2} \log \frac{a}{b}$ **(b)** $F(y) = \frac{1}{y+1}$; $I = \log \frac{a+1}{b+1}$

(c) $F(y) = \frac{y}{y^2+1}$; $I = \frac{1}{2} \frac{b^2+1}{a^2+1}$

(d) $F(y) = \frac{1}{y^2+1}$; $I = \tan^{-1} b - \tan^{-1} a$

(e) $F(y) = \frac{y}{y^2+1}$; $I = \frac{1}{2} \log(1+a^2)$

(f) $F(y) = \frac{1}{y^2+1}$; $I = \tan^{-1} a$

19. (a) $(-1)^n n! (y+1)^{-n-1}$ **(b)** $\pi 2^{-2n-1} \binom{2n}{n} y^{-n-1/2}$

(c) $\frac{n!}{2y^{n+1}} (\log y)^{-2}$ **(d)** $\frac{1}{(\log x)^2}$

22. $\int_{-\infty}^\infty |x^n f(x)| dx < \infty$

24. No; the integral defining F diverges for all y .

31. (a) $\frac{\pi}{2} - \tan^{-1} a$